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Further Studies in Aesthetic Field Theory. II: Equations for $e_{\alpha i}$

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Abstract

In a previous paper (Muraskin, 1973), we obtained a bounded particle in 'aesthetic' field theory. The field equations there are implied by a set of equations for a system of basis vector variables, $e_{\alpha i}$. In this paper, we propose a simpler set of field equations for $e_{\alpha i}$. We find that a bounded particle solution to the equations still appears (as determined by axes runs). The particle appears basically similar to the particle found previously.

1. *Introduction*

In a previous paper (Muraskin, 1973),† we found that our aesthetic-type field theory implied the existence of a bounded particle. We also found that there was no sign of singularities appearing anywhere in our mapping program. The field also became small outside the particle. This latter result is not inconsistent with the natural boundary conditions $\Lambda_{ijk} \to 0$ at infinity.

In our previous work, the field equations were for the quantities Λ_{ijk}. These Λ_{ijk} equations are implied by a set of equations for the quantities $e_{\alpha i}$. In this paper we find a simpler set of $e_{\alpha i}$ equations consistent with our basic ideas. We then study this simpler system of $e_{\alpha i}$ equations using computer techniques.

2. $e_{\alpha i}$ *Equations*

In our previous work our equations were

$$de_{\alpha i} = \Lambda_{mik}\, e_{\alpha m}\, dx_k \tag{2.1}$$

$$\Lambda_{ijk} = e_{\alpha i}\, e_{\beta j}\, e_{\gamma k}\, \Lambda_{\alpha\beta\gamma} \tag{2.2}$$

† A detailed list of references will be found therein.

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$$g_{ij} = e_{\alpha i} e_{\beta j} g_{\alpha\beta} \tag{2.3}$$

$$\Lambda_{\alpha\beta\gamma} = g_{\alpha\beta} \phi_\gamma \delta_{\gamma 0} + g_{\alpha\gamma} \theta_\beta \delta_{\beta 0} + \psi_\alpha g_{\beta\gamma} \delta_{\alpha 0} + A_{\alpha\beta\gamma} \delta_{\alpha 0} \delta_{\beta 0} \delta_{\gamma 0} + B_\sigma \delta_{\sigma 0} \varepsilon_{\sigma\alpha\beta\gamma} \tag{2.4}$$

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.5}$$

(2.4) and (2.5) are invariant under the group $O'(3) \times T$. Combining (2.1) and (2.2) we get field equations for $e_{\alpha i}$

$$\frac{\partial e_{\alpha i}}{\partial x_k} = e_{\alpha m} e_{\sigma m} e_{\beta i} e_{\gamma k} \Lambda_{\sigma\beta\gamma} \tag{2.6}$$

The condition that $e_{\alpha i} \to 0$ at infinity implies that $e_{\alpha i}$ is not orthogonal at all points. That is

$$\begin{aligned} e_{\alpha i} e_{\alpha j} &\neq \delta_{ij} \\ e_{\alpha i} e_{\beta i} &\neq \delta_{\alpha\beta} \end{aligned} \tag{2.7}$$

In our previous work we took $e_{\alpha i}$ to be orthogonal at the origin point.

In formulating the system of equations above we required $e_{\alpha i}$ to be the basic field from which all the other fields are constructed. Thus, the change of all fields would be determined from the manner that the field in question is related to $e_{\alpha i}$.

3. *A Simple Set of $e_{\alpha i}$ Equations*

We recall that our coordinate system is taken to be Cartesian in space and we also bring in a time axis. Only linear coordinate transformations are allowed. We introduce a set of basis vectors $e_{\alpha i}$ which are to be the basic field variables. The index α tells us which basis vector we are dealing with. The change of $e_{\alpha i}$ between neighboring points is again a vector, so it should be expressed as an expansion in terms of the basis vectors. Thus, we have

$$de_{\alpha i} = w_{\alpha\beta}(x) e_{\beta i}(x) \tag{3.1}$$

Now, $w_{\alpha\beta}$ should depend on dx_k, the displacement between the two points. Thus, we define $\Lambda_{\alpha\beta k}$ according to

$$w_{\alpha\beta} = \Lambda_{\alpha\beta k} dx_k \tag{3.2}$$

$\Lambda_{\alpha\beta k}$ is a vector under linear transformations. Thus, we write

$$\Lambda_{\alpha\beta k} = e_{\gamma k} \Lambda_{\alpha\beta\gamma}. \tag{3.3}$$

This gives, from (3.2)

$$w_{\alpha\beta} = e_{\gamma k} dx_k \Lambda_{\alpha\beta\gamma} \tag{3.4}$$

From (3.1) and (3.4) we get

$$\frac{\partial e_{\alpha i}}{\partial x_k} = \Lambda_{\alpha\beta\gamma} e_{\beta i} e_{\gamma k} \tag{3.5}$$

As $e_{\alpha i}$ are to be the sole basic fields, we require that $de_{\alpha i}$ be determined from the basic fields $e_{\alpha i}$, themselves. The simplest way this can be achieved is to require $A_{\alpha\beta\gamma}$ be constant. As in our previous work, we may then require that $A_{\alpha\beta\gamma}$ be invariant under $O'(3)\times T$. Then (3.5) with (2.3), (2.4) and (2.5) constitute our present $g=0$ system of equations.

In our previous work, $e_{\alpha i}$ was orthogonal at the origin. Thus, (3.5) and (2.6) become identical at the origin. However, they differ as we move away from the origin on account of (2.7).

The integrability equations associated with (3.5) are

$$A_{\alpha\beta\gamma}A_{\gamma\lambda\rho} - A_{\alpha\beta\gamma}A_{\gamma\rho\lambda} + A_{\alpha\gamma\lambda}A_{\gamma\beta\rho} - A_{\alpha\gamma\rho}A_{\gamma\beta\lambda} = 0 \qquad (3.6)$$

We do not have an infinite number of conditions to be satisfied at the origin, since (3.6) is satisfied at all points if it is satisfied at one point. Thus, unlike our previous work we do not have an infinite number of integrability equations. We note that equation (3.6) is formally the same in structure as the integrability equations when $g\neq 0$.

Using the $O'(3)\times T$ structure (2.4) and (2.5) we get that (3.6) is satisfied if

$$\begin{aligned}\theta_0 = \phi_0 = A_{000}\\ B_0{}^2 = -\theta_0\psi_0\end{aligned} \qquad (3.7)$$

We have taken in our computer work

$$\theta_0 = \phi_0 = A_{000} = B_0 = -\psi_0 = 1 \qquad (3.8)$$

We obtain an extremum in g_{00} at the origin if

$$e_{0k} = -\frac{e_{\delta k}\,e_{\sigma 0}\,e_{\alpha 0}\,g_{\alpha\beta}\,A_{\beta\sigma\delta}}{e_{\sigma 0}\,e_{\alpha 0}\,g_{\alpha\beta}\,A_{\beta\sigma 0}} \qquad (3.9)$$

with $k=1,2,3$ and the summation over δ is over 1, 2, 3. σ, α, β are summed over 1, 2, 3, 0. To get a maximum or minimum, the quantity $A_{tk}dx_t dx_k$ ($t=1,2,3$, $k=1,2,3$) must be positive or negative definite. A_{tk} is given by

$$\begin{aligned}A_{tk} = e_{\rho t}\,e_{\lambda k}\,e_{\alpha 0}\,e_{\chi 0}[&A_{\sigma\chi\rho}A_{\beta\sigma\lambda}g_{\alpha\beta}\\ &+ A_{\sigma\chi\rho}A_{\beta\alpha\lambda}g_{\sigma\beta} + A_{\sigma\lambda\rho}A_{\beta\chi\sigma}g_{\alpha\beta}]\end{aligned} \qquad (3.10)$$

All the summations in (3.10) are over 1, 2, 3, 0.

A maximum in g_{00} was achieved by choosing $e_{\alpha i}$ at the origin to be

$$\begin{array}{llll} e_{11}=0{\cdot}7 & e_{12}=0{\cdot}62 & e_{13}=0{\cdot}46 & e_{10}=2{\cdot}4\\ e_{21}=-0{\cdot}12 & e_{22}=-0{\cdot}08 & e_{23}=-0{\cdot}14 & e_{20}=0{\cdot}082\\ e_{31}=-0{\cdot}015 & e_{32}=-0{\cdot}097 & e_{33}=-0{\cdot}0111 & e_{30}=0{\cdot}092\\ & & & e_{00}=2{\cdot}0 \end{array} \qquad (3.11)$$

e_{01}, e_{02}, e_{03} were calculated from (3.9).

4. *Discussion*

In this section we would like to make some additional comments concerning the foundation of our approach.

A basic postulate in our program is that the fields are analytic. This assumption implies using the field equations that the field at one point can be expressed entirely in terms of the field at another point by means of a Taylor series expansion that converges. Once integrability has been established, the field is fixed at all points by the field equations in a unique manner once the field is given at one point.

All the equations that we have proposed since our initial paper (Muraskin, 1970) have the property that the change of the field between two nearby points can be expressed entirely in terms of the field at the original point. We note that not all equations that one could propose have the above property. For example, the wave equation

$$(\Box - m^2) f(x) = 0 \tag{4.1}$$

can be expressed as (Hamilton, 1959)

$$f(x') = \frac{1}{c} \int d^3x \left\{ D(x - x') \frac{\partial f(x)}{\partial t} - f(x) \frac{\partial D(x - x')}{\partial t} \right\} \tag{4.2}$$

Thus, the field at x is given in terms of the contribution from an infinite number of points. f and $\partial f / \partial t$ are arbitrary on a hypersurface. Such arbitrariness would be a disadvantage in a basic theory. The wave equation also leads to discontinuities in the second derivative across a wave front. We may look at these discontinuities as an unacceptable property of wave solutions from a fundamental point of view. These discontinuities do not occur in a theory based on analytic fields.

It is not yet proved that our field theory is consistent with the notion of analytic fields. That is, even though we can easily prove local existence of solutions to (3.5) [in the manner of Muraskin (1972)] we have not been able to prove global existence. However, the computer program has not given here, so far, the slightest indication that a singularity may be developing anywhere. Thus, this suggests that global existence may well be satisfied also.

In our previous work we considered the equation

$$de_{\alpha i} = \Lambda_{mik} e_{\alpha m} dx_k \tag{4.3}$$

We may write

$$\Lambda_{mik} = e_{\sigma m} e_{\beta i} e_{\gamma k} \Lambda'_{\sigma\beta\gamma} \tag{4.4}$$

We define $\Lambda_{\alpha\beta\gamma}$ by means of

$$\Lambda_{\alpha\beta\gamma} = e_{\alpha m} e_{\sigma m} \Lambda'_{\sigma\beta\gamma} \tag{4.5}$$

Thus, (4.3) becomes

$$de_{\alpha i} = \Lambda_{\alpha\beta\gamma} e_{\beta i} e_{\gamma k} dx_k \tag{4.6}$$

Thus, equation (3.5) is consistent with the type of equation given by (4.3).

Within our analytic field framework, the field at a point is determined completely from the field at a neighboring point. We thus have the following system of equations:

$$\begin{array}{ll} dA_i = w_{ij}\,dx_k & dB_i = w'_{ij}\,dx_j \\ dw_{ij} = A_{ijk}\,dx_k & dw'_{ij} = A'_{ijk}\,dx_k \\ dA_{ijk} = T_{ijkl}\,dx_l & \vdots \\ \vdots & \vdots \end{array} \tag{4.7}$$

What we are seeking is the simplest system of equations for which the set above closes. That is, we do not wish to introduce a new basic field for the change of each succeeding function within the hierarchy (4.7). We have presented several systems of equations with this closure property since our initial paper (Muraskin, 1970). The equations in this present paper are the simplest system of the type above based on $e_{\alpha i}$ as the basic field.

5. *Computer Results*

We can define A_{ijk} by means of

$$A_{ijk} = e_{\alpha i}\, e_{\beta j}\, e_{\gamma k}\, A_{\alpha\beta\gamma} \tag{5.1}$$

This A_{ijk} will no longer satisfy

$$\frac{\partial A_{ijk}}{\partial x_l} = A_{mjk} A_{mil} + A_{imk} A_{mjl} + A_{ijm} A_{mkl} \tag{5.2}$$

since (3.5) is not the same as (2.6). In our previous work, we had a problem in that so many components of A_{ijk} were repeats of other components. In our present work we still have repeats, but not as many.

We have plotted g_{00} along the x axis in Fig. 1. The plot of g_{00} along y and z has a similar shape. Even though we do not have a turnabout point, along

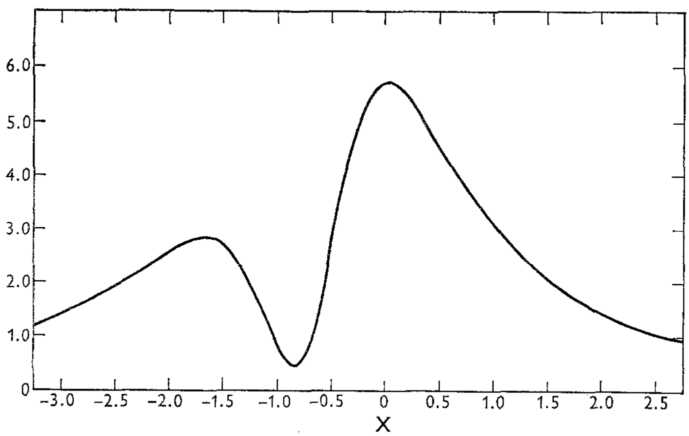

Figure 1.—g_{00} versus x. The grid used was 0·0002.

each of the axes, g_{00} is still nevertheless bounded. g_{00} approaches the bounding value zero in all cases. In view of the absence of a turnabout point in all directions we do not have a natural boundary for the particle associated with the maximum. In addition to the maximum of g_{00} at the origin we have found a minimum in g_{00} in the vicinity of $x=-2{\cdot}64$, $y=0{\cdot}38$, $z=2{\cdot}20$. Here, the value of g_{00} is 0·00008. This leads to the speculation that at the

TABLE 1. g_{00} versus x for large x

x	g_{00}
4·0	0·51
7·0	0·19
10·0	0·10
13·0	0·061
16·0	0·041
19·0	0·029
22·0	0·022
25·0	0·017
30·0	0·012
50·0	0·0044
100·0	0·0011
150·0	0·00050
300·0	0·00012
500·0	0·000045
1000·0	0·000011
1500·0	0·000005
1850·0	0·000003

exact minimum g_{00} may be zero. We have found similar behavior to what appears above in our previous work. Although the present equations differ from the previous ones away from the origin, we did not find any obvious significant differences when making long runs from the origin. In Table 1, we give g_{00} as a function of x for large x. For g_{00}, as well as for all components of $e_{\alpha i}$, we find a monotonic approaching of zero at large distances from the origin. Thus, the qualitative situation is unchanged from our previous work.

We have also investigated other sets of data. We took the following solution of (3.7)

$$\begin{aligned} &\theta_0=\phi_0=A_{000}=0{\cdot}2\\ &B_0=0{\cdot}4\\ &\psi_0=-0{\cdot}8 \end{aligned} \tag{5.3}$$

with $e_{\alpha i}$ still given by (3.11). This gave a maximum in g_{00} at the origin.

A further set of data is given by

$$A_{\alpha\beta\gamma}=\delta_{\alpha\beta}\,\phi_\gamma+\delta_{\alpha\gamma}\,\phi_\beta-\phi_\alpha\,\delta_{\beta\gamma}+\phi_\rho\,\varepsilon_{\rho\alpha\beta\gamma} \tag{5.4}$$

with†

$$\delta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.5}$$

Such a set of data does not satisfy integrability for equation (5.2), but it does for equation (3.5). We chose $\phi_1 = 0{\cdot}2$, $\phi_2 = 0{\cdot}3$, $\phi_3 = 0{\cdot}6$, $\phi_0 = 1{\cdot}0$. We obtain a maximum for g_{00} with $e_{\alpha i}$ again given by (3.10). We note the initial data is unchanged in form under four-dimensional rotations. In both sets of data above, we take $g_{\alpha\beta}$ to be given by (2.5).

In both these cases, we found the g_{00} versus x plots to be similar to Fig. 1.

6. *Summary of Results*

We have obtained the following results:

(1) We get a bounded g_{00} particle (as inferred by axes runs).
(2) There is no sign of singularities appearing anywhere in our computer work.
(3) The fields approach zero far away from the origin in our computer studies.

From selected runs off the axes, we find no evidence contrary to a bound existing in all directions.

Thus, our results are basically similar to those of our previous paper. We see, therefore, that the kind of results we have obtained is not unique to equation (5.2).

Acknowledgement

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† $\delta_{ij} \equiv e_{\alpha i} e_{\beta j} \delta_{\alpha\beta}$ would not have an inverse at each point, since our computer results suggest $e_{\alpha i} \to 0$ at infinity. Thus, we can not raise indices with the inverse of $\delta_{ij}(x)$ at all points.